On similarity-based reasoning

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One of the goals of a variety of approximate reasoning models is to cope with inference patterns more flexible than those of classical reasoning. For instance, we can use terms like *approximately* or *close to* that denote notions of ressemblance or proximity among propositions which need not be fuzzy. One way of proceeding is to equip the set of interpretations (possible worlds) with a fuzzy similarity relation to define the approximation of a proposition. This kind of approach was started by Ruspini [7] and it is followed by Esteva-Godo et al. in some papers [3], [4], [5] where the authors defined two graded consequence relations that model the following reasoning patterns.

One rule like *If A entails B*, where A and B are classical propositions, understood as a logical implication (interpreted as $[A] \subseteq [B]$, where $[X]$ denotes the set of interpretations where X is true), can be modified in the presence of a similarity relation between interpretations, modelling *approximately*, in at least two different ways:

- If A entails B, and we observe A', then it is plausible to conclude approximately B whenever A' is close enough to A (approximate entailment)
- If A entails B, and we observe A', then we can still conclude B whenever A' is close enough to A (strong entailment)

In the first case we are led by the principle that conclusions can also be drawn if they are approximate enough to the correct ones. This leads to a notion of *graded approximate entailment* $\models_a$, that is weaker than the classical one. In the second, we follow the principle that conclusions must remain correct even if the assumption is slightly changed. This leads us to a notion of *graded strong entailment* $\models_a^s$, that is stronger than the classical one.

Given a classical propositional language and a similarity relation between interpretations $S : W \times W \longrightarrow [0,1]$, for each proposition A define $[A]_a = \{w \in W \mid \exists u \in [A]$ such that $S(u,w) \geq a\}$. Then, the approximate and strong entailments can be formally defined as:

- $A \models_a B$ if $[A] \subseteq [B]_a$ (approximate entailment)
- $A \models_a^s B$ if $[A]_a \subseteq [B]$ (strong entailment)

Dedicated to Miguel Delgado in the occasion of his retirement. Excellent researcher, he has been one of the pioneers of fuzzy sets and fuzzy logic in Spain, and founder of the group on fuzzy logic and approximate reasoning at the University of Granada, the largest group in Spain. Along many years, we have jointly participated in many events around the world with Miguel and with our friends from Granada and we have enjoyed his friendship and shared many unforgettable moments. Thanks for all Miguel, and congratulations for this well-deserved homage!

In the cited papers we have studied these two graded entailments from the point of view of their characterizing properties as well as binary modal connectives, and we have provided axiomatizations for them.

In some sense, in the these approaches we have studied inference patterns based on a similarity between possible worlds or interpretations. Worlds where a (classical) proposition is true can be understood as the set of its prototypes.

In [2] we have generalized this type of reasoning to the case of fuzzy propositions. A fuzzy, in the sense of gradual, property is characterized by the existence of borderline cases for which the property only partially applies. We have investigated the case of the most basic description of a vague property α in terms of a set of prototypical situations $[\alpha^+] \subset W$ where α definitely applies, together with a set of counterexamples $[\alpha^-] \subset W$ where α does not apply for sure. We have studied the case of "complete" descriptions where a propert is characterised by its sets of prototypes, its set of counterexamples and the remaining set of situations where we know that α only partially applies to. This model, in some sense, can be fit into the three-valued Łukasiewicz's logic $Ł_3$ set up. While the usual $Ł_3$ logical consequence (based on preservation of truth) takes only care of when the prototypes of premises is a subset of the prototypes of the conclusion, we have shown that the degree-preserving $Ł_3$ logical consequence (see [1]) amounts to also require that the counterexamples of the conclusion be included in those of the premises. Moreover, we have gone one step further by considering a graded notion of entailment, with degrees $a \in [0,1]$, by allowing the prototypes of premises be a-similar to the prototypes of the conclusion, and analogously the counterexamples of the conclusion be a-degree similar to the counterexamples of premises. Thus it is possible to define a graded approximate entailments based either only on prototypes $\models_a$, only on counterexamples $\models_a^C$, or on both examples and counterexamples $\models_a^{\leq}$ defined as

- $\varphi \models_a \psi$ if $[\varphi^+] \subseteq [\psi^+]_a$ (based on prototypes)
- $\varphi \models_a^C \psi$ if $\neg\psi \models_a \neg\varphi$ (based on counterexamples)
- $\varphi \models_a^{\leq} \psi$ iff both $\varphi \models_a \psi$ and $\varphi \models_a^C \psi$ (based on prototypes and counteexamples)

In [2] we have presented a semantical characterization as well as an axiomatization. A very nice report on the limits of the logic of prototypes and counterexamples is [8].

References

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